

ON THE MEAN VALUES OF THE FUNCTION $\tau_k(n)$ IN SEQUENCES OF NATURAL NUMBERS

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ABSTRACT. We obtain an asymptotic formula for the mean value of the function $\tau_k(n)$, which is the number of solutions of the equation $x_1 \cdots x_k = n$ in natural numbers x_1, \dots, x_k in some special sequences of natural numbers.

1. INTRODUCTION

Suppose that $q > 1$, $n = c_0 + c_1q + \dots + c_\nu q^\nu$, $0 \leq c_0, c_1, \dots, c_\nu < q$ — the expansion of the natural number n in the number system of base q . Then $S(n) = c_0 + c_1 + \dots + c_\nu$.

In [1], Gelfond proved the following theorem: *For the number n , $n \leq x$, of integers, satisfying the conditions*

$$n \equiv l \pmod{m}, \quad \sum_{b=0}^{\nu} c_b \equiv a \pmod{p}, \quad n = \sum_{b=0}^{\nu} c_b q^b,$$

where $q > 1$, $p > 1$, $m > 1$; l and a are integers, and $(p, q-1) = 1$, the following asymptotic formula holds:

$$T_0(x) = \frac{x}{mp} + O(x^\lambda), \quad \lambda < 1,$$

where λ is independent of x , m , l , a .

In particular, if $p = q = 2$, then $\lambda = (\ln 3)/(2 \ln 2)$. In this particular case, the author obtained [2] an asymptotic formula for the sum of the form

$$\sum_{\substack{n \leq x \\ \sum c_b \equiv a \pmod{2}}} \tau(n),$$

where a is either 0 or 1.

The proof of Gelfond's theorem is based on his estimate of a trigonometric sum, which we cite in Lemma 1.

In the present paper, we continue studies in this direction. The main result is the following theorem.

Theorem. *Suppose that $k \geq 2$, $p > 1$, and $\varepsilon > 0$ is an arbitrarily small number. Suppose that q is a large natural number such that*

$$\theta(q) = \frac{\ln(6(1 + \ln q))}{\ln q} < \frac{1}{k},$$

here $(q-1, p) = 1$. Suppose that $a \in \mathbb{Z}$ is an arbitrary number.

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Then the following asymptotic formula holds:

$$\sum_{n \leq x; S(n) \equiv a \pmod{p}} \tau_k(n) = \frac{1}{p} \sum_{n \leq x} \tau_k(n) + O\left(x^{1-\frac{1}{k}+\theta(q)+\varepsilon}\right) + O\left(x^{\lambda+\varepsilon}\right),$$

where $\lambda \in (0, 1)$ depends only on p and q .

In order to prove this theorem, in addition to Lemma 1, we need to estimate the integral of the modulus of the trigonometric sum given in Lemma 4.

2. THE LEMMAS

Lemma 1. Suppose that α is an arbitrary real number, $p > 1$, z is an integer, $(z, p) = 1$, $(p, q-1) = 1$, $Q > 1$, and

$$S_Q(\alpha, z) = \sum_{n < q^Q} \exp\left\{\alpha n + \frac{z}{p} S(n)\right\}.$$

Then the following inequality holds:

$$|S_Q(\alpha, z)| \leq q^{\lambda Q},$$

where $\lambda \in (0, 1)$ depends only on p and q .

Proof. For the proof of this lemma, see [1].

Lemma 2. Suppose that T_0 and $T \geq \delta > 0$ are real numbers and, f is a complex-valued continuous function on the closed interval $[T_0, T_0 + T]$ possessing a continuous derivative in the interval $(T_0, T_0 + T)$. Suppose that F is the set of real numbers from the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$ such that $|t - t'| > \delta$ for all distinct numbers t and t' from F . Then

$$\sum_{t \in F} |f(t)| \leq \frac{1}{\delta} \int_{T_0}^{T_0+T} |f(t)| dt + \int_{T_0}^{T_0+T} |f'(t)| dt$$

Proof. For the proof of this lemma, see [3].

Lemma 3. Suppose that $k \geq 2$, $\varepsilon > 0$ is an arbitrarily small number, and $|f(n)| \leq 1$. Then the following inequality holds:

$$\left| \sum_{n \leq x} \tau_k(n) f(n) \right| \ll x^\varepsilon \sum_{l \leq x^{1-\frac{1}{k}}} \left| \sum_{a(l) < ln \leq x} f(ln) \right| + x^{1-\frac{1}{k}+\varepsilon},$$

where $a(l)$ depends only on l and is less than x .

Proof. Suppose that $k = 2$. Let us use the formula

$$(1) \quad \tau(n) = 2 \sum_{0 < l < \sqrt{n}} \frac{1}{l} \sum_{y=0}^{l-1} e^{2\pi i \frac{ny}{l}} + \delta,$$

where $\delta = 1$ or $\delta = 0$ depending on whether n is the square of an integer or not (see [4, p. 53]).

We have

$$\left| \sum_{n \leq x} \tau(n) f(n) \right| \ll \sum_{l < \sqrt{x}} \left| \sum_{l^2 < ln \leq x} f(ln) \right| + \sqrt{x}.$$

Now let $k > 2$. The following identity holds:

$$S = \sum_{n \leq x} \tau_k(n) f(n) = \sum_{mn \leq x} \tau_{k-2}(m) \tau(n) f(mn) = S_1 + S_2,$$

where

$$S_1 = \sum_{mn \leq x; n < (mn)^{2/k}} \tau_{k-2}(m) \tau(n) f(mn),$$

$$S_2 = \sum_{mn \leq x; m \leq (mn)^{1-2/k}} \tau_{k-2}(m) \tau(n) f(mn);$$

here, by definition we set $\tau_1(m)$ identically equal to 1.

Let us estimate the sum S_2 :

$$|S_2| \leq \sum_{m \leq x^{1-2/k}} \tau_{k-2}(m) \left| \sum_{m^{\frac{k}{k-2}} \leq mn \leq x} \tau(n) f(mn) \right|.$$

Again using formula (1), we obtain

$$\begin{aligned} & \sum_{m \leq x^{1-2/k}} \tau_{k-2}(m) \left| \sum_{m^{\frac{k}{k-2}} \leq mn \leq x} \tau(n) f(mn) \right| \leq \\ & \leq 2 \sum_{m \leq x^{1-2/k}} \tau_{k-2}(m) \left| \sum_{m^{\frac{k}{k-2}} \leq mn \leq x} \left(\sum_{d|n; d < \sqrt{n}} 1 \right) f(mn) \right| + \sum_{m \leq x^{1-2/k}} \tau_{k-2}(m) \sum_{n^2 \leq \frac{x}{m}} 1 \ll \\ & \ll \sum_{m \leq x^{1-2/k}} \tau_{k-2}(m) \sum_{d < \sqrt{\frac{x}{m}}} \left| \sum_{\substack{m^{\frac{k}{k-2}} \leq dmn \leq x \\ d^2 m < dmn}} f(dmn) \right| + \sqrt{x} \sum_{m \leq x^{1-2/k}} \frac{\tau_{k-2}(m)}{\sqrt{m}} \ll \\ (2) \quad & \ll \sum_{l < x^{1-1/k}} \left(\sum_{md=l; m \leq x^{1-2/k}; d < \sqrt{\frac{x}{m}}} \tau_{k-2}(m) \right) \left| \sum_{\substack{m^{\frac{k}{k-2}} \leq ln \leq x \\ dl < ln}} f(ln) \right| + x^{1-\frac{1}{k}+\varepsilon}. \end{aligned}$$

Suppose that $l = dm$, where $d < \sqrt{\frac{x}{m}}$, $m \leq x^{1-2/k}$, and $a(l, m) = \max([m^{\frac{k}{k-2}}] + 1, dl)$. Let $a(l)$ — be the value of $a(l, m)$, for which the modulus of the sum

$$\sum_{a(l, m) < ln \leq x} f(ln)$$

is maximal. Then

$$|S_2| \ll \sum_{l < x^{1-1/k}} \left(\sum_{m|l; m \leq x^{1-2/k}; \frac{l}{m} < \sqrt{\frac{x}{m}}} \tau_{k-2}(m) \right) \left| \sum_{a(l) < ln \leq x} f(ln) \right| + x^{1-\frac{1}{k}+\varepsilon} \ll$$

$$\ll x^\varepsilon \sum_{l < x^{1-1/k}} \left| \sum_{a(l) < lm \leq x} f(lm) \right| + x^{1-\frac{1}{k}+\varepsilon}.$$

It remains to estimate the sum S_1 .

If $k = 3$, then $\tau_{k-2}(m) = 1$, $\frac{2}{k} = 1 - \frac{1}{k}$, therefore,

$$|S'_1| \leq \sum_{l < x^{1-1/k}} \tau(l) \sum_{l^{\frac{k}{2}} < lm \leq x} f(lm) \ll x^\varepsilon \sum_{l < x^{1-1/k}} \left| \sum_{l^{\frac{k}{2}} < lm \leq x} f(lm) \right|.$$

If $k = 4$, then $\frac{2}{k} = 1 - \frac{2}{k}$, therefore,

$$|S'_1| \leq \sum_{l < x^{1-2/k}} \tau(l) \sum_{l^{\frac{k}{2}} < lm \leq x} \tau(m) f(lm).$$

The last sum is estimated in the same way as S_2 .

Suppose that $k \geq 5$. The following identity holds:

$$S_1 = S'_1 + S''_1,$$

where

$$\begin{aligned} S'_1 &= \sum_{n < x^{2/k}} \tau(n) \sum_{\substack{n^{\frac{k}{2}} < m_1 m_2 n \leq x \\ m_2 < (m_1 m_2 n)^{2/k}}} \tau_{k-4}(m_1) \tau(m_2) f(m_1 m_2 n), \\ S''_1 &= \sum_{n < x^{2/k}} \tau(n) \sum_{\substack{n^{\frac{k}{2}} < m_1 m_2 n \leq x \\ m_1 n \leq (m_1 m_2 n)^{1-2/k}}} \tau_{k-4}(m_1) \tau(m_2) f(m_1 m_2 n). \end{aligned}$$

For S''_1 we have the inequality

$$|S''_1| \leq \sum_{l \leq x^{1-2/k}} \left(\sum_{\substack{n|l \\ n < x^{2/k}}} \left(\tau(n) \tau_{k-4}\left(\frac{l}{n}\right) \right) \right) \left| \sum_{\substack{n^{k/2} < lm_2 \leq x \\ l^{\frac{k}{k-2}} < lm_2}} \tau(m_2) f(lm_2) \right|.$$

The last sum is estimated in the same way as S_2 .

Consider the sum

$$S'_1 = \sum_{n < x^{2/k}} \tau(n) \sum_{m_2 < x^{2/k}} \tau(m_2) \sum_{\substack{n^{\frac{k}{2}} < m_1 m_2 n \leq x \\ m_2^{k/2} < (m_1 m_2 n)}} \tau_{k-4}(m_1) f(m_1 m_2 n).$$

Note that, as a result of the transformation $\tau_{k-2}(m)$ is replaced by $\tau_{k-4}(m_1)$. If $k \neq 5$, 6 we shall repeat this transformation until $\tau_{k-4}(m_1)$ is replaced by $\tau_1(m_1)$ or $\tau(m_1)$ (depending on whether k is even or odd).

If k is an odd number, then, as a result, we obtain the inequality

$$|S'_1| \ll x^\varepsilon \sum_{l < x^{1-1/k}} \left| \sum_{a(l) < lm \leq x} f(lm) \right|,$$

where $a(l)$ is a number less than x and, possibly, not coinciding with the number $a(l)$ which appears in the estimate of S_2 .

But if k is an even number, then we obtain the inequality

$$|S'_1| \ll x^{\frac{5}{2}} \sum_{l < x^{1-2/k}} \left| \sum_{b(l) < lm \leq x} \tau(m) f(lm) \right|,$$

where $b(l) < x$. The last sum is estimated in the same way as S_2 .

Lemma 3 is proved. \square

Lemma 4. Suppose that $Q > 1$,

$$S_Q(\alpha, z) = \sum_{n < q^Q} e^{2\pi i(\alpha n + \frac{z}{p} S(n))}.$$

Then the following inequality holds:

$$\int_0^1 |S_Q(\alpha, z)| d\alpha \leq q^{Q\theta},$$

where $\theta = \frac{\ln(6(1+\ln q))}{\ln q}$.

Proof. We have the identity

$$S_Q(\alpha, z) = \prod_{r=0}^{Q-1} \sum_{n=0}^{q-1} e^{2\pi i n(\alpha q^r + \frac{z}{p})}$$

Let us divide the interval of integration into q equal parts:

$$\int_0^1 |S_Q(\alpha, z)| d\alpha = \sum_{j=0}^{q-1} \int_{j/q}^{(j+1)/q} |S_Q(\alpha, z)| d\alpha.$$

In all the resulting integrals, let us make the change of variables

$$\alpha = \frac{x+j}{q}, \quad j = 0, 1, \dots, q-1.$$

Then we obtain

$$\int_0^1 |S_Q(\alpha, z)| d\alpha = \frac{1}{q} \sum_{j=0}^{q-1} \int_0^1 \left| S_Q\left(\frac{x+j}{q}, z\right) \right| dx.$$

Consider $S_Q\left(\frac{x+j}{q}, z\right)$. Extracting the first factor in the product, we obtain the identity

$$\begin{aligned} S_Q\left(\frac{x+j}{q}, z\right) &= \sum_{n=0}^{q-1} e^{2\pi i n(\frac{x+j}{q} + \frac{z}{p})} \prod_{r=1}^{Q-1} \sum_{n=0}^{q-1} e^{2\pi i n(\frac{x+j}{q} q^r + \frac{z}{p})} = \\ &= \sum_{n=0}^{q-1} e^{2\pi i n(\frac{x+j}{q} + \frac{z}{p})} S_{Q-1}(x, z). \end{aligned}$$

Suppose that

$$h_j(x) = \left| \sum_{n=0}^{q-1} e^{2\pi i n(\frac{x+j}{q} + \frac{z}{p})} \right|.$$

We have the inequality

$$\int_0^1 |S_Q(\alpha)| d\alpha \leq \int_0^1 \frac{1}{q} \sum_{j=0}^{q-1} h_j(x) |S_{Q-1}(x)| dx.$$

Let us obtain a uniform (in x) estimate of the sum

$$\sum_{j=0}^{q-1} h_j(x).$$

The inequality $h_j(x) \leq q$ is trivial.

Further, if $\frac{z}{p} + \frac{x+j}{q} \notin \mathbb{Z}$, then

$$h_j(x) = \left| \frac{1 - e^{2\pi i q \left(\frac{x+j}{q} + \frac{z}{p} \right)}}{1 - e^{2\pi i \left(\frac{x+j}{q} + \frac{z}{p} \right)}} \right| \leq \frac{1}{\left| \sin \pi \left(\frac{x+j}{q} + \frac{z}{p} \right) \right|} \leq \frac{1}{2 \left\| \frac{x+j}{q} + \frac{z}{p} \right\|},$$

where $\|x\|$ is the distance from x to the nearest integer. Thus,

$$h_j(x) \leq \min \left(q, \frac{1}{2 \left\| \frac{x+j}{q} + \frac{z}{p} \right\|} \right).$$

Let us estimate the sum

$$\frac{1}{q} \sum_{j=0}^{q-1} \min \left(q, \frac{1}{2 \left\| \frac{x+j}{q} + \frac{z}{p} \right\|} \right) \leq \frac{6}{q} (q + q \ln q) = 6(1 + \ln q)$$

(see, for example, [5]).

We have obtained the inequality

$$\int_0^1 |S_Q(\alpha)| d\alpha \leq 6(1 + \ln q) \int_0^1 |S_{Q-1}(\alpha)| d\alpha,$$

valid for any Q , greater than 1. It follows that

$$\begin{aligned} \int_0^1 |S_Q(\alpha)| d\alpha &\leq (6(1 + \ln q))^{Q-1} \int_0^1 |S_1(\alpha)| d\alpha \leq \\ &\leq (6(1 + \ln q))^{Q-1} \int_0^1 \min \left(q, \frac{1}{\left\| \alpha + \frac{z}{p} \right\|} \right) d\alpha. \end{aligned}$$

Let us estimate the last integral. Since the function $\|x\|$ is periodic with period 1 and even, it follows that

$$\int_0^1 \min \left(q, \frac{1}{\left\| \alpha + \frac{z}{p} \right\|} \right) d\alpha = 2 \int_0^{1/2} \min \left(q, \frac{1}{\|\alpha\|} \right) d\alpha \leq 2 + 2 \int_{1/q}^{1/2} \frac{d\alpha}{\alpha} < 2 + 2 \log q.$$

This yields

$$\int_0^1 |S_Q(\alpha)| d\alpha \ll (6(1 + \ln q))^Q = q^{Q \frac{\ln(6(1+\ln q))}{\ln q}},$$

which proves the assertion. □

3. PROOF OF THE THEOREM

1. *Preparing for the application of a large sieve.* We have the chain of equalities

$$\begin{aligned} \sum_{n \leq x; S(n) \equiv a \pmod{p}} \tau_k(n) &= \sum_{n \leq x} \frac{1}{p} \sum_{z=0}^{p-1} \tau_k(n) e^{2\pi i \frac{z(S(n)-a)}{p}} = \\ &= \frac{1}{p} \sum_{n \leq x} \tau_k(n) + \frac{1}{p} \sum_{z=1}^{p-1} e^{-2\pi i \frac{az}{p}} \sum_{n \leq x} \tau_k(n) e^{2\pi i \frac{zS(n)}{p}}. \end{aligned}$$

Now, in order to prove the theorem, it suffices to estimate the sum

$$\sum_{n \leq x} \tau_k(n) e^{2\pi i \frac{zS(n)}{p}}.$$

for any noninteger $\frac{z}{p}$. We apply Lemma 3 to this sum, obtaining:

$$\left| \sum_{n \leq x} \tau_k(n) e^{2\pi i \frac{zS(n)}{p}} \right| \ll W x^\varepsilon + x^{1-\frac{1}{k}+\varepsilon},$$

where

$$W = \sum_{l \leq x^{1-\frac{1}{k}}} \left| \sum_{a(l) < ln \leq x} e^{2\pi i (\frac{z}{p} S(ln))} \right|,$$

$a(l) < x$. Suppose that $H \in \mathbb{N}$, $x \in [q^{H-1}, q^H)$, and $H \geq H_0 > 1$. Then

$$\begin{aligned} W &\leq \sum_{l \leq x^{1-\frac{1}{k}}} \frac{1}{l} \sum_{b=1}^l \left| \sum_{a(l) < n \leq x} e^{2\pi i (\frac{b}{l}n + \frac{z}{p}S(n))} \right| = \\ &= \sum_{l \leq x^{1-\frac{1}{k}}} \frac{1}{l} \sum_{b=1}^l \left| \sum_{n < q^H} e^{2\pi i (\frac{b}{l}n + \frac{z}{p}S(n))} \sum_{a(l) < n_1 \leq x} \int_0^1 e^{2\pi i y(n_1 - n)} dy \right| \leq \\ &\leq \int_0^1 \sum_{l < x^{1-1/k}} \frac{1}{l} \sum_{b=1}^l \left| \sum_{a(l) < n_1 \leq x} e^{2\pi i y n_1} \right| \left| \sum_{n < q^H} e^{2\pi i ((\frac{b}{l}-y)n + \frac{z}{p}S(n))} \right| dy \leq \\ &\leq \int_0^1 \min\left(x, \frac{1}{\|y\|}\right) \sum_{l < x^{1-1/k}} \frac{1}{l} \sum_{b=1}^l \left| \sum_{n < q^H} e^{2\pi i ((\frac{b}{l}-y)n + \frac{z}{p}S(n))} \right| dy. \end{aligned}$$

Let us introduce the notation

$$S_H(\alpha, z) = \sum_{n < q^H} e^{2\pi i (\alpha n + \frac{z}{p}S(n))}.$$

The following identity holds:

$$(3) \quad S_H(\alpha, z) = \prod_{l=0}^{H-1} \sum_{u=0}^{q-1} e^{2\pi i u(\alpha q^l + z/p)}.$$

Let us rewrite the sum

$$\begin{aligned}
\sum_{l < x^{1-1/k}} \frac{1}{l} \sum_{b=1}^l \left| S_H \left(\frac{b}{l} - y, z \right) \right| &= \sum_{d < x^{1-1/k}} \sum_{l \leq x^{1-1/k} \atop (b,l)=d} \frac{1}{l} \sum_{b=1}^l \left| S_H \left(\frac{b}{l} - y, z \right) \right| = \\
&= \sum_{d < x^{1-1/k}} \frac{1}{d} \sum_{l_1 \leq \frac{x^{1-1/k}}{d}} \frac{1}{l_1} \sum_{b_1=1}^{l_1} \left| S_H \left(\frac{b_1}{l_1} - y, z \right) \right| \ll \\
&\ll \log x \sum_{l_1 < x^{1-1/k}} \frac{1}{l_1} \sum_{b_1=1}^{l_1} \left| S_H \left(\frac{b_1}{l_1} - y, z \right) \right|,
\end{aligned}$$

where the prime means that $(b_1, l_1) = 1$.

Further,

$$\begin{aligned}
&\sum_{l_1 < x^{1-1/k}} \frac{1}{l_1} \sum_{b_1=1}^{l_1} \left| S_H \left(\frac{b_1}{l_1} - y, z \right) \right| \ll \\
&\ll \sum_{r \leq (1-\frac{1}{k}) \log_q x + 1} q^{-r+1} \sum_{q^{r-1} \leq l_1 < q^r} \sum_{b_1=1}^{l_1} \left| S_H \left(\frac{b_1}{l_1} - y, z \right) \right| \leq \\
&\leq \sum_{r \leq \frac{H}{2}} q^{-r+1} \sum_{q^{r-1} \leq l_1 < q^r} \sum_{b_1=1}^{l_1} \left| S_H \left(\frac{b_1}{l_1} - y, z \right) \right| + \\
&+ \sum_{\frac{H}{2} < r \leq (1-\frac{1}{k}) \log_q x + 1} q^{-r+1} \sum_{q^{r-1} \leq l_1 < q^r} \sum_{b_1=1}^{l_1} \left| S_H \left(\frac{b_1}{l_1} - y, z \right) \right|.
\end{aligned}$$

For $r < \frac{H}{2}$ we use the identity

$$\left| S_H \left(\frac{b_1}{q_1} - y, z \right) \right| = \left| S_{2r} \left(\frac{b_1}{q_1} - y, z \right) \right| \left| S_{H-2r} \left(\left(\frac{b_1}{q_1} - y \right) q^{2r}, z \right) \right|,$$

which immediately follows from (3), and also use Lemma 1

$$\left| S_{H-2r} \left(\left(\frac{b_1}{q_1} - y \right) q^{2r}, z \right) \right| \ll q^{(H-2r)\lambda+1} \ll x^\lambda q^{-2r\lambda+1},$$

where $0 < \lambda < 1$, which is valid for $\frac{z}{p} \notin \mathbb{Z}$.

Thus,

$$\begin{aligned}
W &\ll x^\lambda \log x \sum_{r \leq \frac{H}{2}} q^{-r-2r\lambda+1} \int_0^1 \min \left(x, \frac{1}{\|y\|} \right) \sum_{q^{r-1} \leq l_1 < q^r} \sum_{b_1=1}^{l_1} \left| S_{2r} \left(\frac{b_1}{l_1} - y, z \right) \right| dy + \\
&+ \log x \int_0^1 \min \left(x, \frac{1}{\|y\|} \right) \sum_{\frac{H}{2} < r \leq (1-\frac{1}{k}) \log_q x + 1} q^{-r+1} \sum_{q^{r-1} \leq l_1 < q^r} \sum_{b_1=1}^{l_1} \left| S_H \left(\frac{b_1}{l_1} - y, z \right) \right| dy.
\end{aligned}$$

2. *Termination of the proof.* Now we have all that is needed to apply the inequality of the large sieve (Lemma 2). We have

$$W \ll x^\lambda \log x \int_0^1 \min \left(x, \frac{1}{\|y\|} \right) dy \sum_{r \leq \frac{H}{2}} q^{-r-2r\lambda+1} \int_0^1 |S_{2r}(\alpha - y, z)| d\alpha +$$

$$+ \log x \int_0^1 \min \left(x, \frac{1}{\|y\|} \right) dy \sum_{\frac{H}{2} < r \leq \left(1 - \frac{1}{k}\right) \log_q x + 1} q^{r+1} \int_0^1 |S_H(\alpha - y, z)| d\alpha.$$

Since the functions $S_{2r}(\alpha, z)$ and $S_H(\alpha, z)$ are periodic with period, by Lemma 4 we have

$$\begin{aligned} \int_0^1 |S_{2r}(\alpha - y, z)| d\alpha &= \int_0^1 |S_{2r}(\alpha, z)| d\alpha \ll q^{2r\theta}, \\ \int_0^1 |S_H(\alpha - y, z)| d\alpha &= \int_0^1 |S_H(\alpha, z)| d\alpha \ll q^{H\theta} \ll x^\theta; \end{aligned}$$

therefore,

$$\begin{aligned} W &\ll x^\lambda \log^2 x \sum_{r \leq \frac{H}{2}} q^{r(1-2\lambda+2\theta)+1} + \log^2 x \sum_{\frac{H}{2} < r \leq \left(1 - \frac{1}{k}\right) \log_q x + 1} q^{r+1} x^\theta \ll \\ &\ll \left(x^\lambda \sum_{r \leq \frac{H}{2}} q^{r(1-2\lambda+2\theta)+1} + qx^{1-\frac{1}{k}+\theta} \right) \log^2 x. \end{aligned}$$

Consider the following two cases:

- 1) if $1 - 2\lambda + 2\theta < 0$; then $\sum_{r \leq \frac{H}{2}} q^{r(1-2\lambda+2\theta)+1} \ll 1$;
- 2) if $1 - 2\lambda + 2\theta \geq 0$; then $\sum_{r \leq \frac{H}{2}} q^{r(1-2\lambda+2\theta)+1} \ll q^{\frac{H}{2}(1-2\lambda+2\theta)+1} \ll x^{\frac{1}{2}-\lambda+\theta}$.

Thus,

$$|W| \ll \left(x^\lambda + x^{1-\frac{1}{k}+\theta} \right) \ln^2 x \ll \left(x^\lambda + x^{1-\frac{1}{k}+\theta} \right) \ln^2 x.$$

(the multiplier q in the final estimate is not written, because we assume that the parameter q is nonincreasing with the growth of the main parameter x).

Theorem is proved.

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